Exam. Code : 211001
Subject Code : 4848

M.Sc. Mathematics 1st Semester

ALGEBRA—I

Paper—MATH-553

Time Allowed—Three Hours] [Maximum Marks—100

Note :— Candidates are required to attempt **FIVE** questions, selecting at least **ONE** question from each section. The fifth question may be attempted from any section.

SECTION—A

1. (a) State and prove Lagrange's theorem and also prove that every group of order prime is cyclic. 10

   (b) Prove that subgroup of a cyclic group is cyclic. 6

   (c) Show that relation of normality is not transitive. 4

2. (a) If G is a group such that G/Z(G) is cyclic where Z(G) is centre of the group G then prove that G is abelian. Also find the cardinality of the centre of a group of order 15. 10

(b) If H and K are finite subgroups of a group G, then prove that $O(HK) = \frac{O(H).O(K)}{O(H \cap K)}$. 7

(c) Give an example to show that union of two subgroups of a group G need not be a subgroup of G. 3

SECTION—B

3. (a) Show that A_n is simple $\forall\ n \geq 5$. 10

(b) Let G be a group and leg G' be the derived group of G. If $H \lhd G$ and G/H is abelian then prove that $G' \subseteq H$. 3

(c) Find all non-isomorphic abelian groups of order 360. 3

(d) If G is a cyclic group of order mn, where (m, n) = 1, then prove that $G \cong H \times K$, where H is a subgroup of order m, and K is a subgroup of order n. 4

4. (a) Prove that the alternating group A_n is generated by the set of all 3-cycles in S_n. 6

(b) Show that $GL(2, \mathbb{R})/SL(2, \mathbb{R}) \cong \mathbb{R}^*$. 6

(c) If H and K both are normal subgroups of a group G such that $H \subseteq K$, then show that :

$$G/K \cong (G/H)/(K/H).$$ 6

(d) Prove that the product of two even permutations is again an even permutation. 2

SECTION—C

5. (a) Prove that subgroup and factor group of a solvable group is again solvable. 6

(b) Show that a simple group is solvable if and only if it is cyclic. 5

(c) State and prove Sylow first theorem. 6

(d) If G is a finite p-group then prove that it has non-trivial centre. 3

6. (a) State and prove Jordan Holder theorem. 10

(b) Write the class equation of Dihedral group of order 8. 5

(c) Let G be a group of order 108. Show that there exists a normal subgroup of order 9 or 27. 5

SECTION—D

7. (a) Prove that an ideal M of a commutative ring R with unity is maximal ideal if and only if R/M is a field. 10

(b) Prove that $12\mathbb{Z}$ is an ideal of the ring $3\mathbb{Z}$ and find all ideals of the ring $3\mathbb{Z}/12\mathbb{Z}$. 5

(c) Is there any integral domain which has 6 elements ? 5

8. (a) Prove that every nilpotent ideal is nil ideal and give an example to show that converse need not be true. 6

(b) Define Boolean ring and let R be a Boolean ring, show that each prime ideal $P \neq R$ is maximal. 6

(c) Let f be a homomorphism of a ring R onto ring R'. Show that $R/\ker f \cong R'$. 6

(d) Let F be a field. Prove that (0) i.e. ideal generated by 0 is a prime ideal in F. 2